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TRIGONOMETRIC RATIOS AND IDENTITIES

SUMMARY OF CONCEPTS

MEASUREMENT OF ANGLES

There are three systems for measurement of an angle:

1. Sexagesimal System or English System In this system an angle is measured in degrees, minutes and seconds. A complete rotation describes 360°.

1 right angle = 90°, (read as 90 degrees)

1° = 60′ (read as 60 minutes)

1′ = 60″ (read as 60 seconds)

Centesimal or French System In this system an gle is measured in grades, minutes and seconds.

1 right angle = 100^g (read as 100 grades)

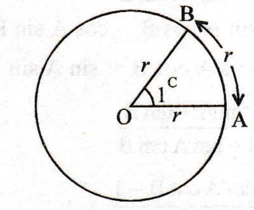
1^g = 100′ (read as 100 minutes)

1′ = 100″ (read as 100 seconds)

Note: 1′ of centesimal system ≠ 1′ of sexagesimal system
1″ of centesimal system ≠ 1″ of sexagesimal system

3. Radian or Circular Measure A radian is a constant angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle and is denoted by 1^c.

∠AOB = 1 radian.



This angle does not depend upon the radius of the circle from which it is derived.

Note: Radian is a unit to measure angle and it should not be inter preted that π stands for 180°, π is a real number whereas π^c stands for 180°.

Remember: π radians = 180° = 200^g.

Relation between Different Systems of Measurement of Angles

$$1^\circ = \frac{10}{9} \text{ grades} ; \ 1^g = \frac{9}{10} \text{ degrees}$$

$$1^\circ = \frac{\pi}{180} \text{ radians} ; \ 1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

$$1^g = \frac{\pi}{200} \text{ radians} ; \ 1 \text{ radian} = \frac{200}{\pi} \text{ grades.}$$

Thus if the measure of an angle in degrees, grades and radians be D, G and θ respectiveiy, then

$$\frac{D}{180} = \frac{G}{200} = \frac{\theta}{\pi}.$$

RELATION BETWEEN SIDES AND INTERIOR ANGLES OF A REGULAR POLYGON

1. Sum of interior angles of polygon of n sides
$= (2n-4) \times 90^\circ$

2. Each interior angle of a regular polygon of n sides
$= \frac{2n-4}{n} \times 90^\circ.$

FUNDAMENTAL IDENTITIES

1. $\sin^2\theta + \cos^2\theta = 1$ or $\cos^2\theta = 1 - \sin^2\theta$
or $\sin^2\theta = 1 - \cos^2\theta$
2. $1 + \tan^2\theta = \sec^2\theta$ or $\sec^2\theta - \tan^2\theta = 1$
3. $1 + \cot^2\theta = \mathrm{cosec}^2\theta$ or $\mathrm{cosec}^2\theta - \cot^2\theta = 1.$

Note: Since $\sin^2\theta + \cos^2\theta = 1$, $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$

$\Rightarrow \quad -1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$;

$0 \le \sin^2\theta \le 1$, $0 \le \cos^2\theta \le 1$.

Since cosec θ = 1/sin θ, cosec θ ≥ 1 or cosec θ ≤ – 1

Also, since sec θ = 1/cos θ, sec θ ≥ 1 or sec θ ≤ – 1.

Sign of Trigonometric Ratios

Trigonometric Ratios \ Quadrants	I	II	III	IV
sin, cosec	+	+	–	–
cos, sec	+	–	–	+
tan, cot	+	–	+	–

Domain and Range of Trigonometric Ratios

Functions	Domain	Range
$\sin x, \cos x$	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	$(-\infty,\infty)-\left\{(2n+1)\frac{\pi}{2}\,\middle\vert\, n\in I\right\}$	$(-\infty, \infty)$
$\cot x$	$(-\infty, \infty)-\{n\pi \mid n\in I\}$	$(-\infty, \infty)$
$\sec x$	$(-\infty,\infty)-\left\{(2n+1)\frac{\pi}{2}\,\middle\vert\, n\in I\right\}$	$(-\infty, -1]\cup[1, \infty)$
$\operatorname{cosec} x$	$(-\infty,\infty)-\{n\pi \mid n\in I\}$	$(-\infty, -1]\cup[1, \infty)$

Trigonometric Ratios of Standard Angles

Angles / T-Ratios	0°	30°	45°	60°	90°
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined
$\operatorname{cosec} x$	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Undefined
$\cot x$	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Trigonometric Ratios of Allied Angles (their sum or Difference is a Multiple of 90°)

	$-\theta$	$90^\circ-\theta$	$90^\circ+\theta$	$180^\circ-\theta$	$180^\circ+\theta$	$270^\circ-\theta$	$270^\circ+\theta$	$360^\circ-\theta$	$360^\circ+\theta$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

Working Rule to Find Allied Angles

Case I: When the angle is $n\pi \pm \theta$, where $n \in I$ and θ is acute.

(a) There is no change in trigonometric function *i.e.* sin remains sin, cos remains cos and tan remains tan. Angle associated becomes θ.

(b) The sign is affixed according to the quadrant in which the angle lies.

Case II: When the angle is $\frac{n\pi}{2} \pm \theta$, where n is an odd integer and θ is acute.

(a) The trigonometric function is replaced by its cofunction i.e. sin changes to cos, tan changes to cot and sec changes to cosec and vice-versa.
Angle associated becomes θ.

(b) The sign is affixed according to the quadrant in which the angle lies.
Note that the sign is always decided on the basis of the operating function.

SOME USEFUL RESULTS ON ALLIED ANGLES

1. $\sin n\pi = 0$, $\cos n\pi = (-1)^n$.
2. $\sin(n\pi+\theta) = (-1)^n\sin\theta$, $\cos(n\pi+\theta) = (-1)^n\cos\theta$
3. $\sin\left(\frac{n\pi}{2}+\theta\right) = \begin{cases}(-1)^{\frac{n-1}{2}}\cos\theta, \text{ if } n \text{ is odd,}\\ (-1)^{\frac{n}{2}}\sin\theta, \text{ if } n \text{ is even.}\end{cases}$

$$\cos\left(\frac{n\pi}{2}+\theta\right) = \begin{cases}(-1)^{\frac{n+1}{2}}\sin\theta, \text{ if } n \text{ is odd,}\\ (-1)^{\frac{n}{2}}\cos\theta, \text{ if } n \text{ is even.}\end{cases}$$

ADDITION AND SUBTRACTION FORMULAE

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
3. $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$
4. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
5. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
6. $\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$
7. $\cot(A+B) = \frac{\cot A\cot B-1}{\cot A+\cot B}$
8. $\cot(A-B) = \frac{\cot A\cot B+1}{\cot B-\cot A}$
9. $\sin(A+B)\sin(A-B) = \sin^2A - \sin^2B$
10. $\cos(A+B)\cos(A-B) = \cos^2A - \sin^2B$
11. $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

or

$= \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$

12. $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

or

$= \cos A\cos B\cos C\,(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$

13. $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

14. $\tan\left(\dfrac{\pi}{4}+A\right) = \dfrac{1+\tan A}{1-\tan A}$

15. $\tan\left(\dfrac{\pi}{4}-A\right) = \dfrac{1-\tan A}{1+\tan A}$.

16. $\sin(A_1 + A_2 + \dots + A_n) = \cos A_1\cos A_2 \dots \cos A_n\,(S_1 - S_3 + S_5 - \dots)$

17. $\cos(A_1 + A_2 + \dots + A_n) = \cos A_1\cos A_2 \dots \cos A_n\,(1 - S_2 + S_4 - S_6 + \dots)$

18. $\tan(A_1 + A_2 + \dots + A_n) = \dfrac{S_1-S_3+S_5-\dots}{1-S_2+S_4-S_6+\dots}$

where $S_1 = \Sigma\tan A_1$, $S_2 = \Sigma\tan A_1\tan A_2$, $S_3 = \Sigma\tan A_1\tan A_2\tan A_3$ and so on.

19. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin(\alpha+(n-1)\beta)$

$$= \frac{\sin\left(\alpha+(n-1)\dfrac{\beta}{2}\right)}{\sin\dfrac{\beta}{2}}\sin\left(\frac{n\beta}{2}\right)$$

20. $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots + \cos(\alpha+(n-1)\beta)$

$$= \frac{\cos\left(\alpha+(n-1)\dfrac{\beta}{2}\right)}{\sin\dfrac{\beta}{2}}\sin\left(\frac{n\beta}{2}\right)$$

TRANSFORMATION FORMULAE

Product into Sum or Difference

1. $2\sin A\cos B = \sin(A+B) + \sin(A-B),\ A > B$
2. $2\cos A\sin B = \sin(A+B) - \sin(A-B),\ A > B$
3. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Sum and Difference into Product

1. $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$
2. $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$
3. $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$
4. $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$
5. $\tan C + \tan D = \dfrac{\sin(C+D)}{\cos C\cos D}$
6. $\tan C - \tan D = \dfrac{\sin(C-D)}{\cos C\cos D}$
7. $\cot C + \cot D = \dfrac{\sin(C+D)}{\sin C\sin D}$
8. $\cot C - \cot D = \dfrac{\sin(D-C)}{\sin C\sin D}$

TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES

(An Angle of the form $n\theta$, $n \in I$)

1. $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$
2. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

   $= 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$
3. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
4. $\cot 2\theta = \dfrac{\cot^2\theta-1}{2\cot\theta}$
5. $1 + \cos 2\theta = 2\cos^2\theta,\ \cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta)$
6. $1 - \cos 2\theta = 2\sin^2\theta,\ \sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta)$
7. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta,\ \sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$
8. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta,\ \cos^3\theta = \dfrac{1}{4}(\cos 3\theta + 3\cos\theta)$
9. $\tan 3\theta = \dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$
10. $\cot 3\theta = \dfrac{\cot^3\theta-3\cot\theta}{3\cot^2\theta-1}$
11. $\cos A\cos 2A\cos 2^2A \dots \cos 2^{n-1}A = \dfrac{\sin 2^n A}{2^n\sin A}$

TRIGONOMETRIC RATIOS OF SUBMULTIPLE ANGLES

(An Angle of the form $\dfrac{\theta}{n}$, $n \in I$)

1. $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = \dfrac{2\tan\theta/2}{1+\tan^2\theta/2}$
2. $\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2}$

   $= \dfrac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$
3. $\tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$
4. $\cot\theta = \dfrac{\cot^2\dfrac{\theta}{2}-1}{2\cot\theta/2}$
5. $\cos^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{2}$
6. $\sin^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{2}$
7. $\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$
8. $\cot^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{1-\cos\theta}$

9. $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$ 10. $\dfrac{1+\cos\theta}{\sin\theta} = \cot\dfrac{\theta}{2}$

TRIGONOMETRIC RATIOS OF SOME SPECIAL ANGLES

1. $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ 2. $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$
3. $\tan 15° = 2-\sqrt{3}$ 4. $\cot 15° = 2+\sqrt{3}$
5. $\sin 22\dfrac{1}{2}° = \dfrac{1}{2}(\sqrt{2-\sqrt{2}})$
6. $\cos 22\dfrac{1}{2}° = \dfrac{1}{2}(\sqrt{2+\sqrt{2}})$ 7. $\tan 22\dfrac{1}{2}° = \sqrt{2}-1$
8. $\cot 22\dfrac{1}{2}° = \sqrt{2}+1$
9. $\sin 18° = \dfrac{\sqrt{5}-1}{4}$ 10. $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$
11. $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$ 12. $\cos 36° = \dfrac{\sqrt{5}+1}{4}$
13. $\sin 9° = \dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4}$
14. $\cos 9° = \dfrac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4}$
15. $\tan 18° = \dfrac{\sqrt{25-10\sqrt{5}}}{5}$
16. $\tan 36° = \sqrt{5-2\sqrt{5}}$

GREATEST AND LEAST VALUES OF THE EXPRESSION

$a \sin\theta + b\cos\theta$

Let $a = r\cos\alpha$, $b = r\sin\alpha$, then

$a^2+b^2 = r^2$ or $r = \sqrt{a^2+b^2}$

Then $a\sin\theta + b\cos\theta = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$
$= r\sin(\theta+\alpha)$

But $-1 \le \sin(\theta+\alpha) \le 1$, so
$-r \le r\sin(\theta+\alpha) \le r$

or $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$.

Thus, the greatest and least values of $a\sin\theta + b\cos\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

SOME USEFUL IDENTITIES

If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$
(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$
(iii) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$
(iv) $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$
(v) $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$
(vi) $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1$.

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following problems:

1. The expression $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A}$ is equal to
(a) $\cot 2A$ (b) $\tan 2A$
(c) $\cot 3A$ (d) $\tan 3A$

2. If $\operatorname{cosec}\theta = x + \dfrac{1}{4x}$ then the value of $\operatorname{cosec}\theta + \cot\theta$ is
(a) $2x$ (b) $-2x$
(c) $\dfrac{1}{2x}$ (d) $-\dfrac{1}{2x}$

3. If $3\sin\theta + 5\cos\theta = 5$, then the value of $5\sin\theta - 3\cos\theta$ is
(a) 3 (b) -3
(c) 5 (d) -5

4. If $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$, then $\cot\alpha\cot\beta\cot\gamma =$
(a) $\cot\delta$ (b) $-\cot\delta$
(c) $\tan\delta$ (d) $-\tan\delta$

5. The value of the expression $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1}$ is
(a) $\dfrac{1-\sin A}{\cos A}$ (b) $\dfrac{1+\sin A}{\cos A}$
(c) $\dfrac{\cos A}{1-\sin A}$ (d) $\dfrac{\cos A}{1+\sin A}$

6. If $\tan^2\theta = 1 - e^2$, then $\sec\theta + \tan^3\theta\operatorname{cosec}\theta =$
(a) $(1-e^2)^{3/2}$ (b) $(2-e^2)^{1/2}$
(c) $(2-e^2)^{3/2}$ (d) None of these
[Based on IIT 1974]

7. The value of the expression $2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$ is
(a) $\sin\alpha$ (b) $\sin 2\alpha$
(c) $\cos\alpha$ (d) $\cos 2\alpha$
[Based on UPSEAT 1993]

8. The value of the expression $2(\sin^6 A + \cos^6 A) - 3(\sin^4 A + \cos^4 A) + 1$ is
(a) 0 (b) 1
(c) -1 (d) None of these
[Based on PET (MP) 1997]